B.Sc. (Hon's) Mathematics (First Semester) Examination-2014

Calculus (Paper Code - AU6859)

Model Solution/Suggestive Answer

1 (i) $D^n[e^x x^2] = D^n e^x . x^2 + {}^nC_1 D^{n-1} e^x Dx^2 + {}^nC_2 D^{n-2} e^x D^2 x^2 + \cdots$

$$= x^2 e^x + 2nxe^x + n(n-1)e^x.$$

(ii). $y = (ax+b)^m$

$y_1 = ma(ax+b)^{m-1}$

$y_2 = m(m-1)a^2(ax+b)^{m-2}$

$\vdots$

$y_n = m(m-1)(m-2)\cdots(m-n+1)a^n(ax+b)^{m-n}$

(iii). Let $f: S \to \mathbb{R}$, $S \subseteq \mathbb{R}$ be a function. Let $c \in S$ be a limit of S. Then f is said to be differentiable at c if

$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c} \quad \text{or} \quad \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$$

exists finitely.

Ex:- $f(x) = x^n$, n being a +ve integer, is differentiable at 0.

(iv). A function f is said to have discontinuity of first kind at $x=a$ if left hand limit i.e. $\lim_{x\to a+0} f(x)$ and right hand limit i.e. $\lim_{x\to a+0} f(x)$ exist but both are not equal.

(v). Let $\epsilon > 0$ be given.

There exists $\delta (= \epsilon)$ s.t.

$$|x-a| < \delta \Rightarrow |f(x)-a| = |x-a| < \epsilon$$

(vi). $f(x) = e^x$

By Taylor's theorem, we have

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots \quad \text{—(1)}$$

Here, we have $f(x) = e^x$ and $a = 1$

Now, $f'(x) = e^x \Rightarrow f'(1) = e$

$f''(x) = e^x \Rightarrow f''(1) = e$

$f'''(x) = e^x \Rightarrow f'''(1) = e$

From (i), we have

$$e^x = e + (x-1)e + \frac{(x-1)^2}{2!}e + \cdots$$

$$= e\left(1 + (x-1) + \frac{(x-1)^2}{2!} + \cdots\right)$$

(vii). $\int_a^b f(x)dx = \lim_{h\to\infty} h[f(a)+f(a+h)+f(a+2h)+\cdots+f\{a+(n-1)h\}]$, $h=\frac{b-a}{n}$

Here $h=\frac{2-1}{n}=\frac{1}{n}$

$$\int_1^2 e^x dx = \lim_{h\to\infty} h\,[e+e^{(1+h)}+e^{1+2h}+\cdots+e^{1+(n-1)h}]$$

$$= \lim_{n\to\infty} h\,[e+e e^{h}+e.e^{2h}+\cdots+e.e^{(n-1)h}]$$

$$= \lim_{h\to 0} he\,[1+e^{h}+e^{2h}+\cdots+e^{(n-1)h}]$$

$$= \lim_{h\to 0} \frac{he.(e^{nh}-1)}{(e^{h}-1)} = \lim_{h\to 0}\frac{he(e-1)}{\{\cancel{1}+h+\frac{h^2}{2!}+\cdots-\cancel{1}\}}$$

$$= e(e-1).$$

(viii). Yes, the function $f(x)=c$ is continuous on $\mathbb{R}$.

2. $y=\cos(m\sin^{-1}x)$

$y_1 = -\sin(m\sin^{-1}x)\times\frac{m}{\sqrt{1-x^2}}$

Squaring both sides and multiplying by $(1-x^2)$, we get

$(1-x^2)y_1^2 = m^2\sin^2(m\sin^{-1}x)$

or $(1-x^2)y_1^2 = m^2(1-y^2)$

Differentiating both sides w.r.t. x, we get

$(1-x^2)2y_1y_2 = 2xy_1^2 - 2m^2yy_1$

or $(1-x^2)y_2 - xy_1 + m^2y = 0.$

Now differentiating every term n times by using Leibnitz's theorem, we get

$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}+(m^2-n^2)y_n=0.$

3. $y=\frac{x}{x^2-a^2}=\frac{1}{2}\left[\frac{1}{(x+a)}+\frac{1}{(x-a)}\right]$ (Resolving into partial fractions)

Differentiating n times, we get

$$y_n = \frac{1}{2}\left[\frac{(-1)^n n!}{(x+a)^{n+1}}+\frac{(-1)^n n!}{(x-a)^{n+1}}\right]$$

$$= \frac{(-1)^n n!}{2}\left[\frac{1}{(x+a)^{n+1}}+\frac{1}{(x-a)^{n+1}}\right].$$

4. L.H.L. at $x=2$ is

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(4x^2-3x) = 10$$

R.H.L. at $x=2$ is

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+}(3x+4) = 10$$

Value of the function at $x=2$ i.e. $f(2)=10$

Since L.H.L = R.H.L = $f(2)$

So the function f is continuous at $x=2$.

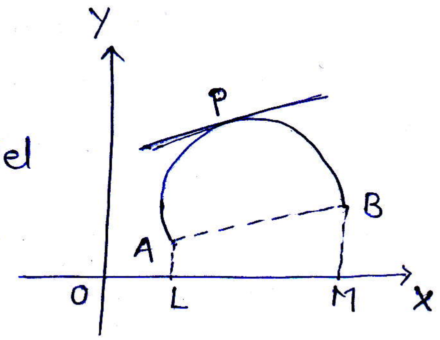
5. Lagrange's Mean Value Theorem :- Let the function $f:[a,b] \longrightarrow \mathbb{R}$ be

(i) continuous on $[a,b]$,

(ii). derivable on (a,b),

then there exists atleast one point $c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$



Geometrically, this theorem states that there is a point P on the curve, the tangent at which is parallel to the chord AB joining the extremities of the curve.

6. $f(x)$, being a polynomial, is continuous on each of closed intervals and is derivable on each of the corresponding open intervals.

So $f(x)$ is continuous on $[-\sqrt{2}, -\frac{1}{2}]$ and differentiable on $(-\sqrt{2}, -\frac{1}{2})$.

$\because f(-\sqrt{2}) = f(-\frac{1}{2}) = 0$

So there exists a point in the interval $(-\sqrt{2}, -\frac{1}{2})$, where $f'(x)=0$.

Now, $f'(x) = 2(x+1)(3x-1) = 0$

gives $x = -1, \frac{2}{3}$.

We observe that $-1 \in (-\sqrt{2}, -\frac{1}{2})$ such that $f'(-1)=0$.

Rolle's theorem is thus verified.

7. $a^2y^2 = x^2(a^2-x^2)$

(i) Symmetry :- This curve is symmetrical about both the axes.

(ii) Origin :- The curve passes through origin.

Tangent at origin: $a^2y^2 - x^2a^2 = 0 \Rightarrow y = \pm x$ (distinct and real)

$\therefore$ origin is a node.

(iii). Asymptotes :- There is no any horizontal asymptote.

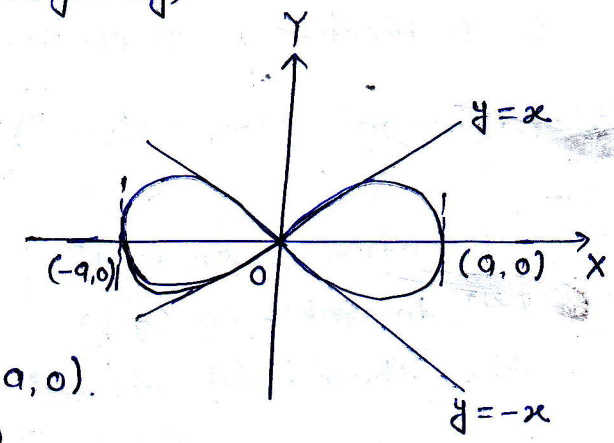
There is no any vertical asymptote.

(iv). Points :– (a). $y = \frac{x}{a}(a^2-x^2)^{1/2}$ (taking +ve sign only)

$$\frac{dy}{dx} = \frac{1}{a}\frac{(a^2-2x^2)}{(a^2-x^2)^{1/2}}$$

$$\frac{dy}{dx} = 0, \text{ when } x = \pm\frac{a}{\sqrt{2}}$$

$$\frac{dy}{dx} = \infty \text{ when } x = \pm a$$



Tangents parallel to y-axis at $(\pm a, 0)$.

(b). curve meets at y-axis at $(0,0)$.
curve meets at x-axis at $(\pm a, 0)$.

(c). If $a^2-x^2<0$ or $x^2-a^2>0$ or $x \in (-\infty,-a) \cup (a,\infty)$
No part of the curve lies in the interval $(-\infty,-a) \cup (a,\infty)$.
so the curve lies only in $(-a,a)$.

8. $\frac{n^{1/2}}{n^{3/2}} + \frac{n^{1/2}}{(n+3)^{3/2}} + \frac{n^{1/2}}{(n+6)^{3/2}} + \cdots + \frac{n^{1/2}}{[n+3(n-1)]^{3/2}}$

The $(r+1)$th term is $\frac{n^{1/2}}{(n+3r)^{3/2}}$ i.e. $\frac{1/n}{(1+3r/n)^{3/2}}$

We, therefore, require the value of

$$\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{1/n}{(1+3r/n)^{3/2}}$$

By the definition of a definite integral as the limit of a sum, we have

$$\int_0^1 \frac{dx}{(1+3x)^{3/2}} = \left[-\frac{2}{3(1+3x)^{1/2}}\right]_0^1 = -\frac{1}{3}+\frac{2}{3} = \frac{1}{3} \text{ Ans.}$$